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UNIFIED CONSTRAINED DYNAMICS

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ABSTRACT

The unified constrained dynamics is formulated without making use of the Dirac splitting of constraint classes. The strengthened, completely-closed, version of the unified constraint algebra generating equations is given. The fundamental phase variable supercommutators are included into the unified algebra as well. The truncated generating operator is defined to be nilpotent in terms of which the Unitarizing Hamiltonian is constructed.

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1 Introduction

In previous papers [1, 2] of the present authors a unified algebraic description of Hamiltonian constraints has been proposed. The main motivation of the description is to avoid explicit splitting [3] of constraints into the first- and second-class ones. With this purpose one should include the fundamental phase variable commutators into the unified constraint algebra as well. Thus we conclude that dynamically-passive ghosts should be assigned to the fundamental phase variables.

A version of generating equations of the unified constraint algebra has also been given in papers [1, 2]. These equations generate all the required commutators indeed, but they appear to be too weak to restrict their natural arbitrariness by the canonical transformations only.

In the present paper we give a strengthened version of generating equations that provides for their arbitrariness to be of the canonical nature certainly. Then we construct the Unitarizing Hamiltonian that governs the unified constrained dynamics in the extended phase space.

Notations and Conventions. As is usual, $\varepsilon(A)$ and $\text{gh}(A)$ stand, respectively, for the Grassmann parity and ghost number of a quantity A . The standard supercommutator is denoted by :

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}(-1)^{\varepsilon(\hat{A})\varepsilon(\hat{B})}.$$

2 Strengthened Version of Generating Equations

Let :

$$\hat{\Gamma}^A, \quad A = 1, \dots, N, \quad \varepsilon(\hat{\Gamma}^A) \equiv \varepsilon_A, \quad \text{gh}(\hat{\Gamma}^A) = 0, \quad (2.1)$$

be an initial set of fundamental phase variable operators.

Following the papers [1, 2], let us assign a dynamically-passive ghost parameter Γ_A^* to each operator $\hat{\Gamma}^A$ of the set (2.1):

$$\hat{\Gamma}^A \longmapsto \Gamma_A^*, \quad \varepsilon(\Gamma_A^*) = \varepsilon_A + 1, \quad \text{gh}(\Gamma_A^*) = 1. \quad (2.2)$$

The dynamical passivity implies that $[\Gamma_A^*, \Gamma_B^*] \equiv 0$ and, besides, no ghost parameters Γ_A^* have their own conjugate momenta.

In turn, let

$$\hat{\Theta}^\alpha(\hat{\Gamma}), \quad \alpha = 1, \dots, M < N, \quad \varepsilon(\hat{\Theta}^\alpha) \equiv \varepsilon_\alpha, \quad (2.3)$$

be a total set of operator-valued irreducible constraints. Let us assign a pair of canonically-conjugated ghost operators to each of the constraints (2.3) :

$$\hat{\Theta}^\alpha(\hat{\Gamma}) \longmapsto (\hat{C}_\alpha, \hat{\mathcal{P}}^\alpha), \quad \alpha = 1, \dots, M, \quad (2.4)$$

$$\varepsilon(\hat{C}_\alpha) = \varepsilon(\hat{\mathcal{P}}^\alpha) = \varepsilon_\alpha + 1, \quad \text{gh}(\hat{C}_\alpha) = -\text{gh}(\hat{\mathcal{P}}^\alpha) = 1, \quad (2.5)$$

$$(i\hbar)^{-1}[\hat{C}_\alpha, \hat{\mathcal{P}}^\beta] = \delta_\alpha^\beta. \quad (2.6)$$

Next, let us introduce the operator functions :

$$\hat{\Omega}(\hat{\Gamma}, \Gamma^*, \hat{C}, \hat{\mathcal{P}}), \quad \hat{\Delta}(\hat{\Gamma}, \Gamma^*, \hat{C}, \hat{\mathcal{P}}), \quad (2.7)$$

$$\hat{\Omega}_\alpha(\hat{\Gamma}, \Gamma^*, \hat{C}, \hat{\mathcal{P}}), \quad \alpha = 1, \dots, M, \quad (2.8)$$

$$\varepsilon(\hat{\Omega}) = 1, \quad \text{gh}(\hat{\Omega}) = 1, \quad \varepsilon(\hat{\Delta}) = 0, \quad \text{gh}(\hat{\Delta}) = 2, \quad (2.9)$$

$$\varepsilon(\hat{\Omega}_\alpha) = \varepsilon_\alpha + 1, \quad \text{gh}(\hat{\Omega}_\alpha) = 1, \quad (2.10)$$

to satisfy the following equations :

$$(i\hbar)^{-1}[\hat{\Omega}, \hat{\Omega}] = \hat{\Delta}, \quad \hat{\Delta}|_{\Gamma^*=0} = 0, \quad (2.11a, b)$$

$$(i\hbar)^{-1}[\hat{\Omega}, \hat{\Omega}_\alpha] = 0, \quad (i\hbar)^{-1}[\hat{\Omega}_\alpha, \hat{\Omega}_\beta] = 0 \quad (2.12a, b)$$

These equations require their compatibility conditions :

$$(i\hbar)^{-1}[\hat{\Omega}, \hat{\Delta}] = 0, \quad (i\hbar)^{-1}[\hat{\Delta}, \hat{\Omega}_\alpha] = 0. \quad (2.13a, b)$$

to be fulfilled.

The previously-given version [1, 2] of generating equations did not include the operators (2.8) so that the equations (2.12) were also absent. Thus, in fact, we have been dealt with the equations (2.11) only. However, being the ghost parameters Γ_A^* different from zero, these equations themselves appear to be insufficient to determine a certain solution for $\hat{\Omega}$, even up to a canonical transformation. To fix such a solution, one needs a given operator $\hat{\Delta}$ satisfying the compatibility condition (2.13a) selfconsistently.

The present, strengthened, version includes the new equations (2.12) that restrict effectively the arbitrariness of the operators $\hat{\Omega}$, $\hat{\Delta}$, $\hat{\Omega}_\alpha$ and thus make it possible just determine a

solution for these operators modulo Γ^* -dependent canonical transformataion in the extended phase space $(2.1) \oplus (2.4)$.

Given the constraints (2.3), one should seek for a solution to the generating equations in the form of a series expansion in powers of the ghost parameters Γ^* and operators $\hat{C}_\alpha, \hat{\mathcal{P}}^\alpha$:

$$\hat{\Omega} = \hat{C}_\alpha \hat{\Theta}^\alpha(\hat{\Gamma}) + \dots, \quad (2.14)$$

$$\hat{\Delta} = -2\hat{C}_\alpha \Gamma_A^* \hat{E}^{A\alpha}(\hat{\Gamma})(-1)^{\varepsilon_\alpha} - \Gamma_B^* \Gamma_A^* \hat{D}^{AB}(\hat{\Gamma})(-1)^{\varepsilon_B} + \dots, \quad (2.15)$$

$$\hat{\Omega}_\alpha = \hat{C}_\beta \hat{\Lambda}_\alpha^\beta(\hat{\Gamma}) + \dots \quad (2.16)$$

Classical counterparts of the corresponding operators should satisfy the following rank conditions on the constraint surface :

$$\text{rank} \|\partial_A \Theta^\alpha(\Gamma)\|_{\Theta=0} = \text{rank} \|\Lambda_\alpha^\beta(\Gamma)\|_{\Theta=0} = M, \quad (2.17)$$

$$\text{rank} \|E^{A\alpha}(\Gamma)\|_{\Theta=0} = M', \quad (2.18)$$

$$\text{corank} \|D^{AB}(\Gamma)\|_{\Theta=0} = M'', \quad M' + M'' = M. \quad (2.19)$$

The conditions (2.18), (2.19) encode the presence of M' first-class and M'' second-class constraints among the M linearly-independent functions $\Theta^\alpha(\Gamma)$. That is the meaning of the functions $E^{A\alpha}(\Gamma)$ and $D^{AB}(\Gamma)$.

As for the functions $\Lambda_\alpha^\beta(\Gamma)$, their meaning is that the linear combinations $\tilde{\Theta}^\alpha \equiv (\Lambda^{-1})^\alpha_\beta \Theta^\beta$ are Abelian constraints. Of course, the inverse matrix Λ^{-1} may appear to be nonlocal in field-theoretic case so that explicit use of the Abelian constraints $\tilde{\Theta}^\alpha$ themselves is rather undesirable in general.

Substituting the expansions (2.14) – (2.16) into the generating equations (2.11), (2.12) and then solving these equations together with their compatibility conditions (2.13) in all orders in ghosts, one obtains in a usual way all the structural relations of the unified constraint algebra. In particular, the fundamental supercommutators $(i\hbar)^{-1}[\hat{\Gamma}^A, \hat{\Gamma}^B]$ and constraint involution relations are generated, respectively, to the $(\Gamma^*)^2$ and $(\hat{C})^2$ -orders of the equation (2.11), as it has been shown in papers [1, 2].

3 Unitarizing Hamiltonian

Having the generating equations (2.11), (2.12) solved for the operators $\hat{\Omega}$, $\hat{\Delta}$, $\hat{\Omega}_\alpha$, we are in a position to construct the Unitarizing Hamiltonian. With this purpose let us first introduce the operator function :

$$\hat{\Phi}(\hat{\Gamma}, \Gamma^*, \hat{C}, \hat{\mathcal{P}}), \quad \varepsilon(\hat{\Phi}) = 0, \quad \text{gh}(\hat{\Phi}) = 0, \quad (3.1)$$

to satisfy the equations :

$$(i\hbar)^{-1}[\hat{\Phi}, \hat{\Omega}_\alpha] = \hat{\Omega}_\alpha, \quad (3.2)$$

$$\hat{\Phi}|_{\Gamma^*=0} = \frac{1}{2}(\hat{C}_\alpha \hat{\mathcal{P}}^\alpha (-1)^{\varepsilon_\alpha} - \hat{\mathcal{P}}^\alpha \hat{C}_\alpha) \equiv \hat{\Phi}_0. \quad (3.3)$$

The operator $\hat{\Phi}_0$ is nothing other but $\hat{C}\hat{\mathcal{P}}$ -contribution to the total ghost number operator.

Let us suppose, that the equations (3.2) are solved for the operator $\hat{\Phi}$ to be searched in the form of a series expansion in powers of the ghost parameters Γ_A^* and operators \hat{C}_α , $\hat{\mathcal{P}}^\alpha$. Then we define the truncated operator $\hat{\Omega}_T$ by the formula :

$$\hat{\Omega}_T \equiv (i\hbar)^{-1}[\hat{\Phi}, \hat{\Omega}], \quad \varepsilon(\hat{\Omega}_T) = 1, \quad \text{gh}(\hat{\Omega}_T) = 1. \quad (3.4)$$

By definition, this operator possesses the properties :

$$(i\hbar)^{-1}[\hat{\Omega}_T, \hat{\Omega}_T] = 0, \quad (i\hbar)^{-1}[\hat{\Phi}, \hat{\Omega}_T] = \hat{\Omega}_T, \quad (3.5a, b)$$

The mentioned properties make it quite natural define the truncated Hamiltonian operator :

$$\hat{H}_T(\hat{\Gamma}, \Gamma^*, \hat{C}, \hat{\mathcal{P}}), \quad \varepsilon(\hat{H}_T) = 0, \quad \text{gh}(\hat{H}_T) = 0, \quad (3.6)$$

to satisfy the equations :

$$(i\hbar)^{-1}[\hat{H}_T, \hat{\Omega}_T] = 0, \quad (i\hbar)^{-1}[\hat{\Phi}, \hat{H}_T] = 0, \quad (3.7)$$

to be solved in the form of a series expansion in powers of the ghost parameters Γ_A^* and operators \hat{C}_α , $\hat{\mathcal{P}}^\alpha$:

$$\hat{H}_T = \hat{H}_0(\hat{\Gamma}) + \dots, \quad (3.8)$$

where \hat{H}_0 is the initial Hamiltonian of the theory.

At the present stage we have to introduce the dynamically-active Lagrangian multipliers and antighosts:

$$(\hat{\lambda}_\alpha, \hat{\pi}^\alpha) \longmapsto (\hat{\mathcal{P}}_\alpha, \hat{\tilde{C}}^\alpha), \quad \alpha = 1, \dots, M, \quad (3.9)$$

$$\varepsilon(\hat{\lambda}_\alpha) = \varepsilon(\hat{\pi}^\alpha) = \varepsilon_\alpha, \quad \varepsilon(\hat{\mathcal{P}}_\alpha) = \varepsilon(\hat{\tilde{C}}^\alpha) = \varepsilon_\alpha + 1, \quad (3.10)$$

$$\text{gh}(\hat{\lambda}_\alpha) = -\text{gh}(\hat{\pi}^\alpha) = 0, \quad \text{gh}(\hat{\mathcal{P}}_\alpha) = -\text{gh}(\hat{C}^\alpha) = 1, \quad (3.11)$$

$$(i\hbar)^{-1}[\hat{\lambda}_\alpha, \hat{\pi}^\beta] = \delta_\alpha^\beta, \quad (i\hbar)^{-1}[\hat{\mathcal{P}}_\alpha, \hat{C}^\beta] = \delta_\alpha^\beta. \quad (3.12)$$

Then we construct the total charge :

$$\hat{Q} = \hat{\Omega}_T + \hat{\mathcal{P}}_\alpha \hat{\pi}^\alpha, \quad (3.13)$$

$$(i\hbar)^{-1}[\hat{Q}, \hat{Q}] = 0, \quad \varepsilon(\hat{Q}) = 1, \quad \text{gh}(\hat{Q}) = 1, \quad (3.14)$$

in terms of which physical operators $\hat{\mathcal{O}}$ and physical states $|\text{Phys}\rangle$ are defined [4] :

$$(i\hbar)^{-1}[\hat{\mathcal{O}}, \hat{Q}] = 0, \quad \hat{Q}|\text{Phys}\rangle = 0. \quad (3.15)$$

Finally, we construct the Unitarizing Hamiltonian of the theory :

$$\hat{H}_{\text{complete}} = \hat{H}_T + (i\hbar)^{-1}[\hat{\Psi}, \hat{Q}], \quad (3.16)$$

$$\varepsilon(\hat{\Psi}) = 1, \quad \text{gh}(\hat{\Psi}) = -1. \quad (3.17)$$

Being the gauge Fermion $\hat{\Psi}$ chosen in the simplest form :

$$\hat{\Psi} = \hat{\chi}_\alpha(\hat{\Gamma})\hat{C}^\alpha + \hat{\lambda}_\alpha\hat{\mathcal{P}}^\alpha, \quad (3.18)$$

classical counterparts of the gauge operators $\hat{\chi}_\alpha$ should satisfy the unitary-limit rank conditions :

$$\text{rank}||\{\Theta^\alpha, \chi_\beta\}|||_{\chi=0, \Theta=0} = M', \quad (3.19)$$

where the Poisson bracket $\{, \}$ is defined to be a classical counterpart of the supercommutator $(i\hbar)^{-1}[\,,\,]$, and M' enters the rank condition (2.18).

In fact, after the nilpotency property (3.5a) is established, the Unitarizing Hamiltonian (3.16) is constructed along the lines of Ref.[5].

As is usual, one can show physical matrix elements of physical operators to be gauge-independent :

$$\delta_{\hat{\Psi}}\langle\text{Phys2}|\hat{\mathcal{O}}\left(\hat{\Xi}(t)\right)|\text{Phys1}\rangle \equiv 0, \quad (3.20)$$

where $\hat{\Xi}(t)$ is a solution to the Heisenberg equations of motion with the Hamiltonian (3.16).

To conclude this Section, the following remark is in order. We have constructed the Unitarizing Hamiltonian at the general position with respect to the ghost parameters Γ_A^* . Of

course, as the ghost parameters are dynamically-passive, one can choose the value $\Gamma_A^* = 0$ to be quite sufficient for all pragmatic aims after the fundamental supercommutators are established. However, it seems to be more geometrically-natural to consider arbitrary values for Γ_A^* as far as the physical content of the theory does not depend on Γ_A^* . That is an aspect of the general ghost-decoupling property of the BFM-construction (see Ref.[6] for a review).

Being the value $\Gamma_A^* = 0$ still chosen, one has at this point :

$$\hat{\Omega}_T|_{\Gamma^*=0} = \hat{\Omega}|_{\Gamma^*=0}. \quad (3.21)$$

By making use of this relation one can confirm directly that in case the rank of fundamental supercommutators is constant, the Hamiltonian (3.16) certainly reproduces, at the functional integral level, the standard BFM description in which the constraints are split into the first-class and second-class ones [7].

4 General Structure of Generating Operators

Let us now return to the generating equations (2.11), (2.12) to study their natural arbitrariness and transformation properties.

First of all, it follows from the equations (2.12b) that

$$\hat{\Omega}_\alpha = \hat{\mathcal{U}}^{-1} \hat{C}_\alpha \hat{\mathcal{U}}, \quad \varepsilon(\hat{\mathcal{U}}) = 0, \quad \text{gh}(\hat{\mathcal{U}}) = 0, \quad (4.1)$$

where $\hat{\mathcal{U}}(\hat{\Gamma}, \Gamma^*, \hat{C}, \hat{\mathcal{P}})$ is an arbitrary canonical transformation.

Then the equations (2.12a) yield :

$$\hat{\Omega} = \hat{\mathcal{U}}^{-1} (\hat{C}_\alpha \hat{\Theta}^\alpha(\hat{\Gamma}) + \Gamma_A^* \hat{\Gamma}^A) \hat{\mathcal{U}}, \quad (4.2)$$

where $\hat{\Gamma}^A(\hat{\Gamma})$ is a ghost-independent reparametrization of the fundamental phase variable operators.

Due to the equations (2.11) we have :

$$(i\hbar)^{-1} [\hat{\Theta}^\alpha, \hat{\Theta}^\beta] = 0, \quad (4.3)$$

$$\hat{\Delta} = \hat{\mathcal{U}}^{-1} (-2\hat{C}_\alpha \Gamma_A^* (i\hbar)^{-1} [\hat{\Gamma}^A, \hat{\Theta}^\alpha] (-1)^{\varepsilon_\alpha} - \Gamma_B^* \Gamma_A^* (i\hbar)^{-1} [\hat{\Gamma}^A, \hat{\Gamma}^B] (-1)^{\varepsilon_B}) \hat{\mathcal{U}}. \quad (4.4)$$

Thus we conclude that the operators $\hat{\Theta}^\alpha$ are Abelian constraints.

Next, it follows from the equations (3.2), (3.3), (4.1) that

$$\hat{\Phi} = \hat{\mathcal{U}}^{-1} \hat{\Phi}_0 \hat{\mathcal{U}}, \quad (4.5)$$

and hence :

$$\hat{\Omega}_T = \hat{\mathcal{U}}^{-1} \hat{C}_\alpha \hat{\hat{\Theta}}^\alpha \hat{\mathcal{U}}. \quad (4.6)$$

Finally, the equations (3.7), (4.6) yield :

$$\hat{H}_T = \hat{\mathcal{U}}^{-1} \hat{\hat{H}}_T \hat{\mathcal{U}}, \quad (4.7)$$

$$(i\hbar)^{-1}[\hat{C}_\alpha \hat{\hat{\Theta}}^\alpha, \hat{\hat{H}}_T] = 0, \quad (i\hbar)^{-1}[\hat{\Phi}_0, \hat{\hat{H}}_T] = 0. \quad (4.8a, b)$$

It follows from the equation (4.8b) that the operator $\hat{\hat{H}}_T$ does not depend on Γ_A^* . As for the equation (4.8a), its natural arbitrariness is :

$$\hat{\hat{H}}_T \rightarrow \hat{\hat{H}}_T + (i\hbar)^{-1}[\hat{K}, \hat{C}_\alpha \hat{\hat{\Theta}}^\alpha], \quad (4.9)$$

$$\varepsilon(\hat{K}) = 1, \quad (i\hbar)^{-1}[\hat{\Phi}_0, \hat{K}] = -\hat{K}. \quad (4.10)$$

By choosing in the expression (3.16) :

$$\hat{\Psi} = \hat{\mathcal{U}}^{-1} \hat{\hat{\Psi}} \hat{\mathcal{U}}, \quad (4.11)$$

where the operator $\hat{\hat{\Psi}}$ is Γ^* -independent, we arrive at the following representation :

$$\hat{H}_{complete} = \hat{\mathcal{U}}^{-1} \hat{\hat{H}}_{complete} \hat{\mathcal{U}}, \quad (4.12)$$

$$\hat{H}_{complete} \equiv \hat{\hat{H}} + (i\hbar)^{-1}[\hat{\hat{\Psi}}, (\hat{C}_\alpha \hat{\hat{\Theta}}^\alpha + \hat{\mathcal{P}}_\alpha \hat{\pi}^\alpha)]. \quad (4.13)$$

Thus we conclude that the Γ^* -dependence of the Unitarizing Hamiltonian is absorbed totally into the overall operator $\hat{\mathcal{U}}$ of an arbitrary canonical transformation.

5 Conclusion

So, we have formulated the unified constrained dynamics without making use of the Dirac splitting of constraint classes. The selfconsistency of the fundamental supercommutators is guaranteed by including them into the unified constrained algebra. The corresponding algebra-generating equations are shown to be able to determine their solution effectively up to a canonical transformation in the extended phase space. The truncated generating operator is then defined to be nilpotent, in terms of which the Unitarizing Hamiltonian is constructed.

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